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# Note

# An Interesting Set of Simultaneous, Nonlinear Equations

## Introduction

Figure 1 shows a typical oceanographic situation in which two-layer flow occurs between two water bodies connected by a channel which contains a sill.

The equations which describe this situation can be written in normalized form as:

$$u_1^2/y_1 + u_2^2/y_2 = 1 \tag{1}$$

$$y_1 + y_2 = 1 \tag{2}$$

$$u_r^2 = 1 \tag{3}$$

$$u_1 y_1 + u_2 y_2 = U_0 \tag{4}$$

$$u_1 y_1 = u_r y_r B \tag{5}$$

$$y_r + u_r^2/2 = y_1 + (u_1^2 - u_2^2)/2. \tag{6}$$

Here $B$ is the width ratio and $U_0$ is defined to be the barotropic, or total, flow. The layer depth and velocity variables, $y$ and $u$, are defined in Fig. 1a.

## Methods of Solution

Some algebraic manipulation of Eqs. (1)–(6) easily eliminates four of the variables and produces:

$$u_1^2(1-y_1) + y_1(2y_1 + u_1^2 - 3(u_1 y_1/B)^{2/3}) - y_1(1-y_1) = 0$$

and

$$(U_0 - u_1 y_1)^2 - (1-y_1)^2(2y_1 + u_1^2 - 3(u_1 y_1/b)^{2/3}) = 0.$$

A straightforward application of a 2-dimensional Newton–Raphson process yields the curves shown in Fig. 2. The right-hand end points of the curves, obtained by this method and marked with a "—", are quite irregular. Physically, the curves should continue to meet the $U_0$ axis, their failure to do so is caused by round-off noise in the numerical process.



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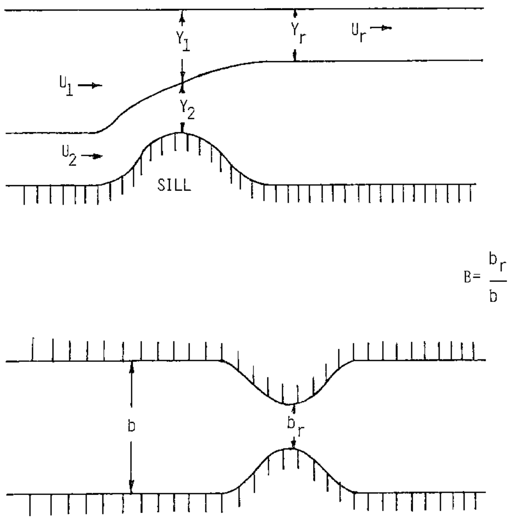


FIG. 1. Vertical section and plan view of a sill with constriction.

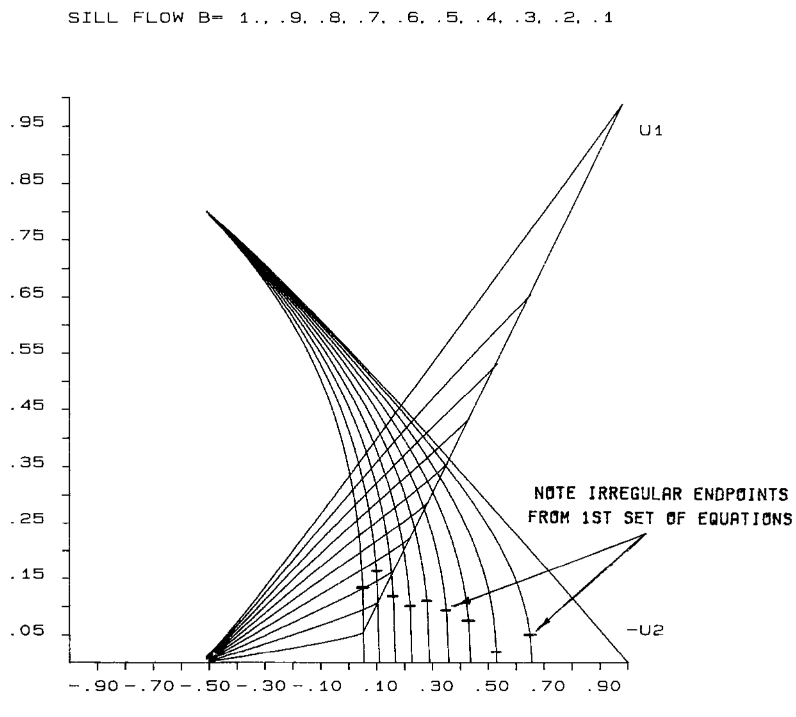


FIG. 2. $u_1$ and $u_2$ vs. $U_0$ for a range of values of $B$. The — marks the end of that portion of the $u_2$ curve obtainable with the first set of equations.

Consider the somewhat simpler eliminants:

$$u_1^2 - 3y_1^{5/3}(u_1/B)^{2/3} + y_1(3y_1 - 1) = 0 \quad (7)$$

and

$$u_1^2/y_1 + (U_0 - u_1 y_1)^2/(1 - y_1)^3 = 1.$$

The last equation easily reduces to a quadratic whose roots are

$$u_1 = [U_0 y_1^2 \pm (1 - y_1)\{y_1(1 - y_1)(1 - 3y_1 + 3y_1^2 - U_0^2)\}^{1/2}]/(1 - 3y_1 + 3y_1^2). \quad (8)$$

## Investigation of the Roots

Equation (7) is easily solved numerically. However, to investigate root locations, the loci of $u_1$ vs. $y_1$ from (7) and (8) for $U_0$ in the range $-1$ to $+1$ and for $B$ in the range 0.1 to $+1$ are plotted. These loci (Fig. 3) reveal the richness of the root structure. The curves for negative $U_0$ are simply the reflections of those for positive $U_0$ in the $y_1$ axis so that, to avoid confusion, the reflections of the $B = \text{const.}$ curves in the same axis have been plotted. The two points of confluence of the $B = \text{const.}$ curves at (0, 0) and (0, $\frac{1}{3}$) are clearly shown.

## Generation of $u_1$ and $u_2$ Curves

Oceanographers are interested in curves (Fig. 2) relating $u_1$ and $u_2$ to $U_0$ for constant values of $B$. To produce these curves compute, for each $U_0$ and $B$ in the range, starting with $y_1 = -1$:

1. $u_1$ from (8) for $y_1$,
2. the value of (7),
3. $y_1$ increased by a suitable interval $dy_1$.

When the value of (7) changes sign, a root had been trapped between the two values of $y_1$. The binary partition method [3] is then used to define $y_1$ to the required accuracy.

The root $y_1$ being found, the corresponding value of $u_1$ is plotted as well as the value of $u_2$ derived from

$$u_2 = (U_0 - u_1 y_1)/(1 - y_1). \quad (9)$$

The resulting curves are shown in Fig. 2. Because $U_0$ proceeds in steps the actual

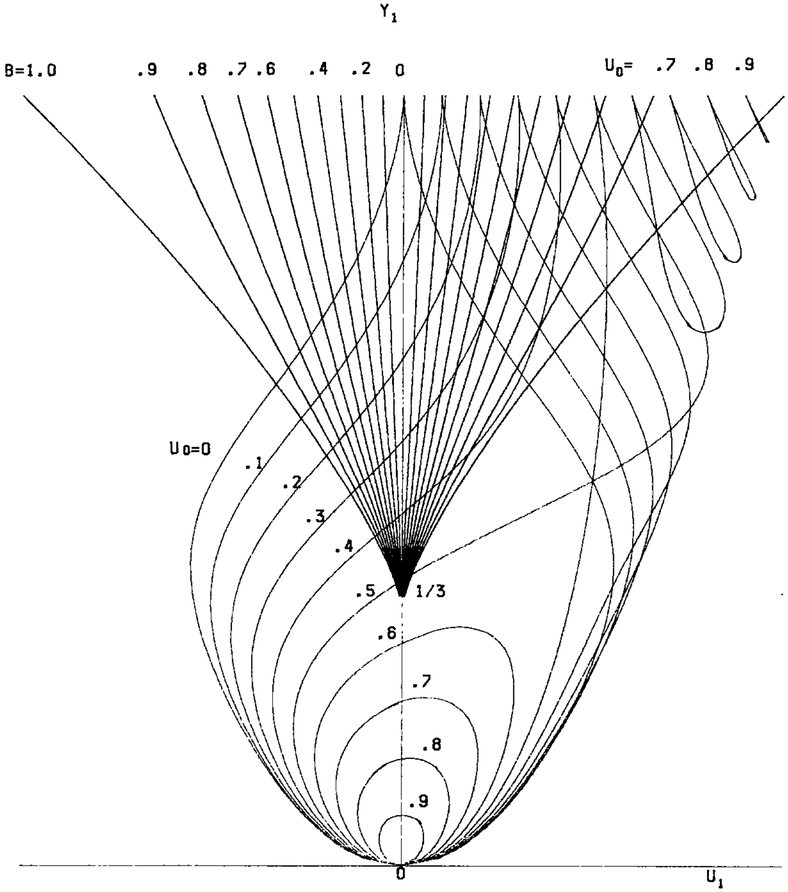


FIG. 3. $B$ = constant and $U_0$ = constant loci. The intersections give the positions of the roots.

end points for $u_2$ on the $U_0$ axis will not, generally, appear. To complete the curves the relationship:

$$B = U_0/((2 + U_0^2)/3)^{3/2},$$

easily derived from (2), (4), (5), and (6) when $y_2 = 0$ defines the end points for positive $U_0$. For negative $U_0$ the confluent points have $y_1 = \frac{1}{3}$ from (7) and thus $U_0 = (\frac{2}{3})^{3/2}$ from (7) and $u_2 = (\frac{2}{3})^{1/2}$ from (9).

The robust binary partition method can be replaced easily by a more rapidly convergent one if desired. A program that is well suited to operation on any conventional microcomputer which has a curve plotter and runs BASIC is available from the authors.

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